Math 249 Lecture 18 Notes

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1 Wrapping up RSK

From last time, we had RSK takes a permutation and give you a pair of SYT with the same shape as the cycle structure of your permutation. This algorithm had an inverse, showing a bijection. We defined the descent set D so that $D(P) = D(\sigma^{-1})$ and $D(Q) = D(\sigma)$. We used pairs (σ, a) last time, but since we have pairs of words, we will add another piece of information: a string of numbers b, like we had before with a.

The bijection is weight preserving, so $x^a = x^S$ and $x^b = x^T$, where we send the triple $(\sigma, a, b) \mapsto (S, T)$.

1.1 From multisets of pairs of numbers to triples (σ, a, b)

We now have to show that these triples are in bijection with the desired multisets of pairs of positive integers. Take the pairs of positive integers (with possible repeats), and sort them lexicographically.

$$\begin{aligned} &2\;5\;1\;3\;3\;3\;3\;4 \\ &1\;1\;2\;2\;2\;4\;5\;5 = b \end{aligned}$$

The second row will be b. We can get a by arranging the first row in increasing order, and we can get σ by mandating that the first row be $\sigma(a) = a \circ \sigma^{-1}$, except we rewrite the row as distinct numbers in increasing order.

$$a = 1\;2\;3\;3\;3\;3\;4\;5$$

$$\sigma = 2\;8\;1\;3\;4\;5\;6\;7$$

Given σ, a, b, can we reverse this process? The condition for b being compatible with σ is that whenever $\sigma(a)$ is constant, b is weakly increasing. This is exactly lexicographic ordering. So we have the desired bijection. This completes the RSK algorithm.

1.2 Example and consequences of RSK

Let's look at RSK directly in action, using the previous multiset of pairs. We get the pair of semistandard young tableau

$$\begin{ytableau} 2 & 5 \\ 1 & 3 & 3 & 3 & 3 & 4 \end{ytableau} \qquad \begin{ytableau} 2 & 2 \\ 1 & 1 & 2 & 4 & 5 & 5 \end{ytableau}$$

Try it out yourself to see that you get the same result!

The RSK algorithm gives us that the generating functions for both these objects are the same:

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x) s_\lambda(y)$$

We could have used this interpretation to define the Schur functions as the number of SSYT of shape λ. Since these two are equal, we get that the Schur functions are indeed symmetric.

2 Littlewood-Richardson coefficients

Definition 2.1. The *Littlewood-Richardson coefficients* are the coefficients in the equation

$$s_\kappa s_\lambda = \sum_{\mu} c^{\mu}_{\kappa,\lambda} s_\mu.$$

2.1 Ways to count the Littlewood-Richardson coefficients

Proposition 2.1. *The Littlewood-Richardson coefficients are nonnegative integers.*

We will give 3 proofs.

2.1.1 GL_N character proof

Proof. You may recall that from the theory of Lie groups, we said that $s_\lambda(x_1, \ldots, x_N)$ is the character of an irreducible module V_λ. So $\chi_{V_\kappa \otimes V_\lambda} = s_\kappa s_\lambda$, and since GL_N modules split as a sum of irreducible modules (with characters s_λ), we get $V_\kappa \otimes V_\lambda \cong \bigoplus V_\mu^{c^{\mu}_{\kappa,\lambda}}$. $\square$

This means we can interpret Littlewood-Richardson coefficients as the multiplicities of GL_N modules.

2.1.2 S_n character proof

Proposition 2.2. *Let χ be a character of S_k, and let φ be a character of S_ℓ. Then*

$$F(\chi)F(\varphi) = F((\chi \otimes \varphi) \uparrow_{S_k \times S_\ell}^{S_{k+\ell}}),$$

where $(\chi \otimes \varphi)(\sigma, \tau) = \chi(\sigma)\varphi(\tau)$ is an $S_k \times S_\ell \subseteq S_{k+\ell}$ character.

Proof. There is a straightforward proof just following the definitions, but it is tedious, so instead, we will cheat. Take $\chi = \mathbb{1} \uparrow_{S_\kappa}^{S_k}$ and $\varphi = \mathbb{1} \uparrow_{S_\lambda}^{S_\ell}$, where κ is a partition of k and λ is a partition of ℓ. [note to the reader: why is it sufficient to just take these two? let me know if you find out] Then $\chi \otimes \varphi = \mathbb{1} \uparrow_{S_\kappa \times S_\lambda}^{S_k \times S_\ell}$. So

$$F((\chi \otimes \varphi) \uparrow_{S_k \times S_\ell}^{S_{k+\ell}}) = F(\mathbb{1} \uparrow_{S_\kappa \times S_\lambda}^{S_{k+\ell}}) = h_\kappa h_\lambda = F(\chi)F(\varphi).$$ □

Our proof now becomes very simple.

Proof. The Schur functions S_μ are the irreducible characters of the symmetric group. So

$$s_\kappa s_\lambda = F(\chi_\kappa)F(\chi_\lambda) = F((\chi_\kappa \otimes \chi_\lambda) \uparrow_{S_k \times S_\ell}^{S_{k+n}}) = \sum c_{\kappa,\lambda}^{\mu} s_\mu$$

because this splits as a sum of irreducible characters. This interpretation of $c_{\kappa,\lambda}^{\mu}$ is the number of μ in the product, so it must be a nonnegative integer. □

2.1.3 Young tableau proof

We will introduce this now and explain next lecture. You can take a skew diagram with shape $\nu = \mu/\lambda$, and define $s_\nu(x) = \sum_{S \in SSYT(\nu)} X^S$ as a generalization of Schur functions. Then $s_\nu = \sum_\kappa c_{\kappa,\lambda}^{\mu} s_\kappa$.